The author thanks the participants of the seminar guided by V.V.Rumiantsev for discussing the paper.

REFERENCES

1. Malkin, I. G., Theory of Stability of Motion. Moscow, "Nauka", 1966.
2. Levi-Cività, T., Sopra alcuni criteri da instabilità. Ann. di Mat. Pura et Appl., Ser. 3, Vol. 5, 1901.
3. Mustakhishev, K. M., On the stability of Hamiltonian systems. Trudy Univ. Druzhby Narodov, Vol. 27, №5, 1968.
4. Markeev, A. P., On the stability of a nonautonomous Hamiltonian system with two degrees of freedom. PMM Vol. 33, №3, 1969.
5. Kamenkov, G. V., Selected Works, Vol. 2, Moscow, "Nauka", 1971.
6. Liapunov, A. M., General Problem of the Stability of Motion. Collected Works, Vol. 2, Moscow, Izd. Akad. Nauk SSSR, 1956.
7. Kamenkov, G. V., Selected Works, Vol. 1, Moscow, "Nauka", 1971.
8. Kunitsyn, A. L., On stability in the critical case of pure imaginary roots under internal resonance. Differents. Uravneniia, Vol. 7, №9, 1971.
9. Kunitsyn, A. L., Stability in the critical case of three pairs of pure imaginary roots in the presence of internal resonance. PMM Vol. 35, №1, 1971.
10. Khazin, L. G., On the stability of Hamiltonian systems in the presence of resonances. PMM Vol. 35, №3, 1971.

Translated by N.H.C.

UDC 531.3

ON THE STABILITY OF NONLINEAR SYSTEMS OF THE NEUTRAL TYPE

PMM Vol. 39, №1, 1975, pp. 45-52
V.P.SKRIPNIK
(Moscow)
(Received May 3, 1973)

We prove that from the stability (asymptotic stability) of linear system (1) follows the stability (respectively, asymptotic stability) of the trivial solution of nonlinear system (2) if the deviations of the arguments and the nonlinear addition are sufficiently small in the correspinding integral sense.

For $l = 1, 2, \ldots, q$ we denote $f(t, \xi_l, \eta_l) = f(t, \xi_1, \xi_2, \ldots, \xi_q, \eta_1, \eta_2, \ldots, \eta_q)$, where f, ξ_l, η_l are m-dimensional vectors. We consider the following two systems: the linear system (1) and the nonlinear system (2) perturbed [1] relative to (1)

$$y' = A(t)\,y, \qquad A(t) = \sum_{k=1}^{p} A_k(t) \tag{1}$$

$$x'(t) = \sum_{k=1}^{p} A_k(t)\, x(\varphi_k(t)) + f(t, x'(\psi_l(t)), x(\chi_l(t))) \tag{2}$$

Here φ_k, ψ_l, χ_l are transformations of the argument, $A_k(t)$ are square matrices, x and y are mth-order vectors. Everywhere the integrals are to be understood in the Lebesgue sense. The derivative is to be understood in the following sense. If for some constant vector c

$$v(t)=c+\int_a^t u(\tau)\,d\tau$$

for $t\in[a, b]$ then $u(t)$ is called the derivative of $v(t)$ and denoted $v'(t)$ for $t\in[a, b]$. In this paper we shall use the following notation: $\|\ \|$ is the norm of a vector or a matrix, equalling the sum of the moduli of the elements; $Y(t)$ is the matrix solution of system (1), satisfying the initial condition $Y(t_0)=E$, where E is the unit matrix; $Z(\theta)$ is a set of m-dimensional vector-valued functions, defined for $t\leqslant\theta$, whose components have an at most countable set of discontinuity points; $W(\theta)$ is a set of m-dimensional vector-valued functions, defined for $t\leqslant\theta$, whose components are continuous.

Let $z\in Z(a)$, $w\in W(a)$. A vector-valued function $x(t)$ satisfying system (2) for $t\in[a, b]$ and the conditions $x'(t)=z(t)$, $x(t)=w(t)$ for $t\leqslant a$ is called a solution of system (2) for $t\in[a, b]$, corresponding to the initial vector-valued functions $z(t)$ and $w(t)$. We note that here we do not require the fulfillment of the so-called connection condition. The problem is to determine the stability conditions for the trivial solution of system (2) if the trivial solution of system (1) is stable.

The following conditions for system (2) are in aggregate called conditions ω.

1) The elements of matrices $A_k(t)$ are determined for $t\in[t_0, \infty)$ and have an at most countable set of discontinuity points.

2) $\|A_k(t)\|\leqslant a_k$ for $t\in[t_0, \infty)$.

3) The vector-valued function $f(t, \xi_l, \eta_l)$ is defined for $t\in[t_0,\infty)$ for $\|\eta_l\|\leqslant R$, where $R>0$, and for any ξ_l.

4) For each component of the vector-valued function $f(t, \xi_l, \eta_l)$ there exists an at most countable set of values of t, at which it suffers a discontinuity.

5) There hold the inequalities

$$\|f(t,\xi_l'',\eta_l'')-f(t,\xi_l',\eta_l')\|\leqslant\sum_{l=1}^{q}G_l\|\xi_l''-\xi_l'\|+\sum_{l=1}^{q}H_l\|\eta_l''-\eta_l'\|$$

$$\|f(t,\xi_l,\eta_l)\leqslant\sum_{l=1}^{q}g_l(t)\|\xi_l\|+\sum_{l=1}^{q}h_l(t)\|\eta_l\|$$

where $g_l(t)$ and $h_l(t)$ are integrable functions on any finite interval $[t_0, T]$ (we can assume that $g_l(t)\leqslant G_l$, $h_l(t)\leqslant H_l$), and

$$\sum_{l=1}^{q}G_l<1,\quad \int_{t_0}^{\infty}g_l(\tau)\,d\tau<\infty,\quad \int_{t_0}^{\infty}h_l(\tau)\,d\tau<\infty$$

6) The functions $\varphi_k(t)$, $\psi_l(t)$ and $\chi_l(t)$ are defined for $t\in[t_0, \infty)$, satisfy the inequalities $\varphi_k(t)$, $\psi_l(t)$, $\chi_l(t)\leqslant t$ and have an at most countable set of discontinuity points.

7) There exists $\tau_0\geqslant t_0$ such that $\varphi_k(t)\geqslant t_0$ for $t\geqslant\tau_0$.

8) The functions $\Delta_k(t)=t-\varphi_k(t)$ are integrable on any finite interval $[t_0, T]$ and

$$\int_{t_0}^{\infty}\Delta_k(\tau)\,d\tau<\infty$$

9) For each function $\psi_l(t)$ any finite interval $[t_0, T]$ can be represented as a

finite sum of intervals on each of which the function either strictly increases or strictly decreases or is constant.

As the sets of initial vector-valued functions we take the sets $Z(t_0)$ and $W(t_0)$. The trivial sultion of system (2) is said to be stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the solution of system (2), corresponding to any initial vector-valued functions $z \in Z(t_0)$, $w \in W(t_0)$ such that $\| z(t) \| < \delta$, $\| w(t) \| < \delta$ for $t \geqslant t_0$, satisfies the inequalities $\| x'(t) \| < \varepsilon$, $\| x(t) \| < \varepsilon$. If $\lim_{t\to\infty} x(t) = 0$, then the trivial solution of system (2) is said to be partially asymptotically stable. However, if $\lim_{t\to\infty} x'(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$, then the trivial solution of system (2) is said to be asymptotically stable.

Theorem 1. Assume that: 1) conditions ω are fulfilled for system (2) ; 2) the matrix $A(t)$ is periodic or is such that

$$\int_{t_0}^{t} \operatorname{sp} A(\tau)\, d\tau \geqslant \alpha > -\infty \tag{3}$$

Then the stability of the trivial solution of system (2) follows from the stability of the trivial solution of system (1).

Proof. We denote $F(t, \xi_l, \eta_l) = f(t, \xi_l, \zeta_l)$, where

$$\zeta_l = \begin{cases} \eta_l, & \|\eta_l\| \leqslant R \\ \dfrac{R}{\|\eta_l\|}\eta_l, & \|\eta_l\| > R \end{cases}$$

The vector-valued function $F(t, \xi_l, \eta_l)$ satisfies the following conditions. It is defined for $t \in [t_0, \infty)$ and for any ξ_l and η_l; for each of its components there exists an at most countable set of values t for which the component becomes discontinuous; for any $\xi_l, \eta_l, \xi_l', \eta_l', \xi_l'', \eta_l''$ we have the inequalities

$$\|F(t, \xi_l, \eta_l\| \leqslant \sum_{l=1}^{q} g_l(t)\|\xi_l\| + \sum_{l=1}^{q} h_l(t)\|\eta_l\|$$

$$\|F(t, \xi_l'', \eta_l'') - F(t, \xi_l', \eta_l')\| \leqslant \sum_{l=1}^{q} G_l\|\xi_l'' - \xi_l'\| + 2\sum_{l=1}^{q} H_l\|\eta_l'' - \eta_l''\|$$

Let us consider the system

$$x'(t) = \sum_{k=1}^{p} A_k(t)\, x(\varphi_k(t)) + F(t, x'(\psi_l(t)), x(\chi_l(t))) \tag{4}$$

We take any sets $Z(\theta_0)$ and $W(\theta_0)$, where $\theta_0 \geqslant t_0$, and any bounded vector-valued functions $z \in Z(\theta_0)$ and $w \in W(\theta_0)$. We denote

$$\mu = \max\{1 + d + (a + H)(\theta_1 - \theta_0),\ a + H\}$$

$$\beta = \sup_{t \leqslant \theta_0}\|z(t)\| + \sup_{t \leqslant \theta_0}\|w(t)\|$$

$$d = \sum_{l=1}^{q} G_l, \qquad H = \sum_{l=1}^{q} H_l, \qquad a = \sum_{k=1}^{p} a_k$$

We take d_0 satisfying the inequality $d < d_0 < 1$. Let θ_1 be such that

$$\theta_1 - \theta_0 = (d_0 - d)/(a + 2H)$$

With the aid of the successive approximations

$$u_n(t)=\begin{cases}\sum_{k=1}^{p} A_k(t)v_{n-1}(\varphi_k(t))+F(t,u_{n-1}(\psi_l(t)),v_{n-1}(\chi_l(t))), & t\in(\theta_0,\theta_1]\\ z(t), & t\leqslant\theta_0\end{cases}$$

$$v_{n-1}(t)=\begin{cases}\omega(\theta_0)+\int_{\theta_0}^{t}u_{n-1}(\tau)\,d\tau, & t\in(\theta_0,\theta_1]\\ w(t), & t\leqslant\theta_0\end{cases}$$

$$u_0(t)=\begin{cases}z(\theta_0), & t\in(\theta_0,\theta_1]\\ z(t), & t\leqslant\theta_0\end{cases}$$

where for $t\in[\theta_0,\theta_1]$ components of vector functions $u_n(t)$ have an at most countable set of discontinuity points, it is not difficult to show that a solution $x(t)$ of system (4) exists for $t\in[\theta_0,\theta_1]$, corresponding to $z(t)$ and $w(t)$ This solution is unique since by assuming the contrary we get that $d_0\geqslant 1$. The inequalities

$$\|x'(t)\|\leqslant\beta+\frac{\mu\beta}{1-d_0},\quad \|x(t)\|\leqslant\beta+\left(\beta+\frac{\mu\beta}{1-d_0}\right)(\theta_1-\theta_0) \tag{5}$$

are valid for $t\in[\theta_0,\theta_1]$.

Let $z\in Z(t_0)$ and $w\in W(t_0)$. By virtue of what we have proved, a unique solution $x(t)$ of system (4) exists for $t\in[t_0,t_1]$, where

$$t_1-t_0=(d_0-d)/a+2H$$

corresponding to the initial vector-valued functions $z(t)$ and $w(t)$. This solution satisfies bounds (5) for $t\in[t_0,t_1]$. Assuming the vector-valued functions $x'(t)$ and $x(t)$ as initial ones, we continue the solution in unique manner on the interval $[t_1,t_2]$, where

$$t_2-t_1=(d_0-d)/a+2H$$

This continuation satisfies bounds of type (5) for $t\in[t_1,t_2]$, etc. Hence it follows that a unique infinitely-continuable solution $x(t)$ of system (4) exists, and it corresponds to the initial vector-valued functions $z(t)$ and $w(t)$. For arbitrary numbers $T>t_0$ and $\varepsilon>0$, we can find $\delta>0$ such that for $t\in[t_0,T]$ we have $\|x'(t)\|<\varepsilon$, $\|x(t)\|<\varepsilon$ for any solution $x(t)$ of system (4), corresponding to the initial vector-valued functions $z\in Z(t_0)$ and $w\in W(t_0)$ such that $\|z(t)\|<\delta$, $\|w(t)\|<\delta$.

When condition (3) is fulfilled, the matrix $Y^{-1}(t)$ is bounded for $t\in[t_0,\infty)$. Therefore, in the given case the quantity $\|Y(t)Y^{-1}(\tau)\|$ is bounded for $t\in[t_0,\infty)$, $\tau\in[t_0,t]$. When the matrix $A(t)$ is periodic, $Y(t)=P(t)e^{Bt}$ [2], where $P(t)$ is a periodic matrix, B is a constant matrix, and e^{Bt} is a bounded matrix. By virtue of the equality $Y(t)Y^{-1}(\tau)=P(t)e^{B(t-\tau)}P^{-1}(\tau)$ the quantity $\|Y(t)Y^{-1}(\tau)\|$ is bounded for $t\in[t_0,\infty)$, $\tau\in[t_0,t]$. Therefore, in any case we can find a number μ_0 such that $\|Y(t)Y^{-1}(\tau)\|\leqslant\mu_0$ for $t\in[t_0,\infty)$, $\tau\in[t_0,t]$.

Let $T_0\geqslant\tau_0$ be such that

$$\mu_0\sum_{k=1}^{p}\int_{T_0}^{\infty}\left(M\Delta_k(\tau)+\mu_0\frac{a+H}{1-d}\sum_{l=1}^{q}\int_{T_0}^{\infty}g_l(\tau)\,d\tau+\mu_0\sum_{l=1}^{q}\int_{T_0}^{\infty}h_l(\tau)\,d\tau\leqslant\theta<1\right.$$

$$M=a^2+ad\frac{a+H}{1-d}+aH$$

We take any $\delta > 0$. Let $z \in Z(t_0)$ and $w \in W(t_0)$ be such that $\| z(t) \| < \delta$ and $\| w(t) \| < \delta$, and let $x(t)$ be the solution of system (4) corresponding to them. We take any $T > T_0$. We denote

$$\delta_1 = \sup_{t \leqslant T_0} \| x'(t) \|, \quad \lambda_1 = \sup_{t \in [T_0, T]} \| x'(t) \|, \quad \delta_2 = \sup_{t \leqslant T_0} \| x(t) \|, \quad \lambda_2 = \sup_{t \in [T_0, T]} \| x(t) \|$$

From equality (4) it follows that

$$\| x'(t) \| \leqslant a(\delta_2 + \lambda_2) + d(\delta_1 + \lambda_1) + H(\delta_2 + \lambda_2)$$

for $t \in [T_0, T]$. Hence we get that

$$\lambda_1 \leqslant \frac{(a+H)(\delta_2 + \lambda_2) + d\delta_1}{1-d} \tag{6}$$

From the equalities

$$x(\varphi_k(t)) - x(t) = \int_t^{\varphi_k(t)} \sum_{k=1}^{p} A_k(\tau)\, x(\varphi_k(\tau))\, d\tau + \int_t^{\varphi_k(t)} F(\tau, x'(\psi_l(\tau)), x(\chi_l(\tau)))\, d\tau$$

valid for $t \geqslant T_0$, taking (6) into account we obtain

$$\| x(\varphi_k(t)) - x(t) \| \leqslant a\Delta_k(t)(\lambda_2 + \delta_2) + d\Delta_k(t)(\lambda_1 + \delta_1) + H\Delta_k(t)(\lambda_2 + \delta_2) \leqslant a\Delta_k(t)\lambda_2 + a\Delta_k(t)\delta_2 + d\Delta_k(t)\frac{(a+H)\lambda_2}{1-d} + d\Delta_k(t)\frac{(a+H)\delta_2 + d\delta_1}{1-d} + d\Delta_k(t)\delta_1 + H\Delta_k(t)\lambda_2 + H\Delta_k(t)\delta_2$$

for $t \in [T_0, T]$. Taking the last inequalities into account we obtain the following estimates when $t \in [T_0, T]$:

$$\| A_k(t)[x(\varphi_k(t)) - x(t)] \| \leqslant M\Delta_k(t) + L_1(\delta_1, \delta_2)\Delta_k(t) \tag{7}$$

$$L_1(\delta_1, \delta_2) = a^2\delta_2 + ad\frac{(a+H)\delta_2 + d\delta_1}{1-d} + ad\delta_1 + aH\delta_2$$

When $t \geqslant t_0$ the solution $x(t)$ satisfies the following system of integral equations:

$$x(t) = y(t) + \int_{t_0}^{t} Y(t)Y^{-1}(\tau) \sum_{k=1}^{p} A_k(\tau)[x(\varphi_k(\tau)) - x(\tau)]\, d\tau + \int_{t_0}^{t} Y(t)Y^{-1}(\tau) F(\tau, x'(\psi_l(\tau)), x(\chi_l(\tau)))\, d\tau \tag{8}$$

where $y(t)$ is the solution of system (1), satisfying the initial condition $y(t_0) = w(t_0)$. Let us separate the integration interval in equality (8) into $[t_0, T_0]$ and $[T_0, t]$. Then, taking inequalities (6) and (7) into account, we obtain the following inequality when $t \in [T_0, T]$:

$$\| x(t) \| \leqslant \delta_0 + L_2(\delta_1, \delta_2) + \lambda_2\theta, \qquad \delta_0 = \sup_{t \geqslant t_0} \| y(t) \|$$

$$L_2(\delta_1, \delta_2) = \mu_0(T_0 - t_0)[2ab_2 + d\delta_1 + H\delta_2] + \mu_0 L_1(\delta_1, \delta_2) \times \sum_{k=1}^{p} \int_{T_0}^{\infty} \Delta_k(\tau)\, d\tau + \mu_0\Big[\delta_1 + \frac{(a+H)\delta_2 + d\delta_1}{1-d}\Big] \sum_{l=1}^{q} \int_{T_0}^{\infty} g_l(\tau)\, d\tau + \mu_0\delta_2 \sum_{l=1}^{q} \int_{T_0}^{\infty} h_l(\tau)\, d\tau$$

Hence

$$\lambda_2 \leqslant \gamma, \qquad \gamma = \frac{\delta_0 + L_2(\delta_1, \delta_2)}{1 - \theta}$$

The latter inequality is independent of T. Therefore, when $t \geqslant T_0$ the vector-valued function $x(t)$ satisfies the following inequality:

$$\| x(t) \| \leqslant \gamma \tag{9}$$

From inequalities (6) and (9) it follows that

$$\| x'(t) \| \leqslant \frac{(a + H)(\delta_2 + \gamma) + d\delta_1}{1 - d}$$

for $t \geqslant T_0$. Since the numbers δ_0, δ_1 and δ_2 can be made arbitrarily small, the trivial solution of system (4) is stable for all $z \in Z(t_0)$ and $w \in W(t_0)$ such that $\| z(t) \| < \delta$ and $\| w(t) \| < \delta$, if δ is sufficiently small. Therefore the trivial solution of system (2) is stable. Theorem 1 is proved.

Theorem 2. Assume that: 1) conditions ω are fulfilled; 2) matrix $A(t)$ is periodic. Then the partial asymptotic stability of the trivial solution of system (2) follows from the asymptotic stability of the trivial solution of system (1).

Proof. From the fact that the trivial solution of system (1) is asymptotically stable it follows, on the basis of Theorem 1, that the trivial solution of system (2) is stable.

Let r satisfy the inequality $0 < r < R$. We can find $\delta > 0$ such that if only the initial vector-valued functions $z \in Z(t_0)$ and $w \in W(t_0)$ satisfy the inequalities $\| z(t) \| < \delta$, $\| w(t) \| < \delta$, then the solution $x(t)$ of system (2) $t \in [t_0, \infty)$ corresponding to them satisfies the inequalities $\| x'(t) \| < r$, $\| x(t) \| < r$. We shall consider such solutions only.

We take any $\varepsilon > 0$. Let $T_0 \geqslant \tau_0$ be such that

$$\mu_0 r \sum_{k=1}^{p} a(a + d + H) \int_{T_0}^{\infty} \Delta_k(\tau)\, d\tau + \mu_0 r \sum_{l=1}^{q} \left(\int_{T_0}^{\infty} g_l(\tau)\, d\tau + \int_{T_0}^{\infty} h_l(\tau)\, d\tau \right) < \frac{\varepsilon}{2}$$

$$\mu_0 = \sup_{t \in [t_0, \infty),\ \tau \in [t_0, t]} \| Y(t) Y^{-1}(\tau) \|$$

Let $T \geqslant T_0$ be such that for $t \geqslant T_0$ we have

$$\| y(t) \| + 2ar \| Y(t) \| \int_{t_0}^{T_0} \| Y^{-1}(\tau) \| d\tau + r(d + H) \| Y(t) \| \int_{t_0}^{T_0} \| Y^{-1}(\tau) \| d\tau < \frac{\varepsilon}{2}$$

where $y(t)$ is the solution of system (1), satisfying the initial condition $y(t_0) = w(t_0)$. We make use of equality (8), valid for $t \geqslant t_0$, in which instead of F there occurs the function f of those same arguments. Taking into account that

$$\| A_k(t) (x(\varphi_k(t)) - x(t)) \| \leqslant ar(a + d + H) \Delta_k(t)$$

for $t \geqslant T_0$, we obtain the following inequality:

$$\| x(t) \| \leqslant \| y(t) \| + 2ar \| Y(t) \| \int_{t_0}^{T_0} \| Y^{-1}(\tau) \| d\tau + \mu_0 r \sum_{k=1}^{p} a(a + d + H) \times \int_{T_0}^{t} \Delta_k(\tau)\, d\tau +$$

$$r(d + H) \| Y(t) \| \int_{t_0}^{T_0} \| Y^{-1}(\tau) \| d\tau + \mu_0 r \left(\sum_{l=1}^{q} \int_{T_0}^{t} g_l(\tau)\, d\tau + \sum_{l=1}^{q} \int_{T_0}^{t} h_l(\tau)\, d\tau \right)$$

which is valid for $t \geqslant T_0$. Therefore, $\| x(t) \| < \varepsilon$ for $t \geqslant T$. But this signifies that $\lim_{t\to\infty} x(t) = 0$. Theorem 2 is proved.

Theorem 3. Assume that: 1) the hypotheses of Theorem 2 are fulfilled; 2) $\lim \varphi_k(t) = \infty$, $\lim \psi_l(t) = \infty$, $\lim \chi_l(t) = \infty$ as $t \to \infty$. Then the asymptotic stability of the trivial solution of system (2) follows from the asymptotic stability of the trivial solution of system (1).

Proof. Let r satisfy the inequality $0 < r < R$. We take $\delta > 0$ such that if only the initial vector-valued functions $z \in Z(t_0)$ and $w \in W(t_0)$ satisfy the inequalities $\| z(t) \| < \delta$, $\| w(t) \| < \delta$, then the corresponding solution $x(t)$ of system (2) satisfies the inequalities $\| x'(t) \| < r$, $\| x(t) \| < r$ when $t \geqslant t_0$ and, in addition, $\lim_{t\to\infty} x(t) = 0$. We shall examine such solutions only; let $x(t)$ be one of them. We take any $\varepsilon > 0$. We take $T_0 \geqslant t_0$ such that

$$\sum_{k=1}^{p} a_k \| x(\varphi_k(t)) \| + \sum_{l=1}^{q} H_l \| x(\chi_l(t)) \| < \varepsilon$$

for $t \geqslant T_0$.

There exists $T(T_0) \geqslant T_0$ such that $\psi_l(t) \geqslant T_0$ when $t \geqslant T(T_0)$. For any $\sigma \geqslant t_0$ we denote $\Delta(\sigma) = \sup_{t\geqslant\sigma} \| x'(t) \|$. Obviously, the function $\Delta(\sigma)$ is nonincreasing. Since it is bounded from below, $\Delta_\infty = \lim_{\sigma\to\infty} \Delta(\sigma)$. exists. From equality (2) we obtain

$$\| x'(t) \| \leqslant \sum_{l=1}^{q} G_l \| x'(\psi_l(t)) \| + \sum_{k=1}^{p} a_k \| x(\varphi_k(t)) \| + \sum_{l=1}^{q} H_l \| x(\chi_l(t)) \| < d\Delta(T_0) + \varepsilon$$

for $t \geqslant T(T_0)$. Hence

$$\Delta(T(T_0)) < d\Delta(T_0) + \varepsilon$$

Passing to the limit as $T_0 \to \infty$, we obtain

$$\Delta_\infty \leqslant d\Delta_\infty + \varepsilon, \qquad \Delta_\infty \leqslant \varepsilon/(1-d)$$

Since ε is arbitrary, $\Delta_\infty = 0$. But this signifies that $\lim_{t\to\infty} x'(t) = 0$. Theorem 3 is proved.

If $d < 1/2$, in conditions ω it is sufficient to require that the vector-valued function $f(t, \xi_l, \eta_l)$ be bounded only for $\| \xi_l \| \leqslant R$. The results of this paper generalize to the case of an infinite countable number of transformed arguments.

Together with system (1) let us consider a system (2) in which $p = \infty$ and $l = 1, 2, 3\ldots$ The following conditions for this system are in aggregate called conditions ω_1:

1) Conditions 1) – 4) and 6) – 9) of conditions ω are fulfilled and

$$a = \sum_{k=1}^{\infty} a_k < \infty, \qquad \sum_{k=1}^{\infty} \int_{t_0}^{\infty} \Delta_k(\tau)\, d\tau < \infty$$

2) There hold the inequalities

$$\| f(t, \xi_l'', \eta_l'') - f(t, \xi_l', \eta_l') \| \leqslant \sum_{l=1}^{\infty} G_l \| \xi_l'' - \xi_l' \| + \sum_{l=1}^{\infty} H_l \| \eta_l'' - \eta_l' \|$$

$$\| f(t, \xi_l, \eta_l) \| \leqslant \sum_{l=1}^{\infty} g_l(t) \| \xi_l \| + \sum_{l=1}^{\infty} h_l(t) \| \eta_l \|$$

where $g_l(t)$ and $h_l(t)$ are integrable functions on any finite interval $[t_0, T]$ (we can assume that $g_l(t) \leqslant G_l$, $h_l(t) \leqslant H_l$), and

$$\sum_{l=1}^{\infty} G_l < 1, \quad \sum_{l=1}^{\infty} H_l < \infty, \quad \sum_{l=1}^{\infty} \int_{t_0}^{\infty} g_l(\tau)\, d\tau < \infty, \quad \sum_{l=1}^{\infty} \int_{t_0}^{\infty} h_l(\tau)\, d\tau < \infty$$

Theorems 1 – 3 will hold for this case if conditions ω are replaced by conditions ω_1.

REFERENCES

1. Krasovskii, N. N., Some Problems in the Theory of Stability of Motion. Moscow, Fizmatgiz, 1959. (See also English translation, Krasovskii, N. N., Stability of Motion, Stanford University Press, Stanford, California, 1963).
2. Bellman, R., Stability Theory of Differential Equations. New York, McGraw-Hill Book Co., Inc., 1953.
3. El'sgol'ts, L. E. and Norkin, S. B., Introduction to the Theory of Differential Equations with a Deviating Argument. Moscow, "Nauka", 1971.

Translated by N. H. C.

UDC 531.36

ON STABILIZATION OF POTENTIAL SYSTEMS

PMM Vol. 39, №1, 1975, pp. 53-58

V. M. LAKHADANOV
(Minsk)

(Received April 11, 1974)

We prove a generalization of the Kelvin-Chetaev theorem. We examine certain aspects of the stabilization of unstable potential systems by gyroscopic and nonconservative forces [1].

1. We consider the systems (D, F are constant symmetric ($n \times n$)-matrices)

$$x^{\cdot\cdot} + Dx^{\cdot} + Fx = 0 \tag{1.1}$$

$$x^{\cdot\cdot} + Dx^{\cdot} + Fx = X(x, x^{\cdot}) \tag{1.2}$$

$$x = \operatorname{col}(x_1, \ldots, x_n),\ X(x, x^{\cdot}) = \operatorname{col}(X_1(x, x^{\cdot}), \ldots, X_n(x, x^{\cdot})),\ X(0,0) \equiv 0$$

(the functions $X_i(x, x^{\cdot})$ contain x_i, $x_i^{\cdot}$ to powers not less than second).

A result which we can state as the following theorem was proved in [2].

The Kelvin-Chetaev theorem. If matrix D is positive definite and among the eigenvalues of matrix F there is at least one negative, then systems (1.1) and (1.2) are unstable.

A result which can be looked upon as a generalization of the Kelvin-Chetaev theorem was proved in [3].

The theorem from [3]. If matrix D is positive definite and $|F| \neq 0$, then the number of roots with a positive real part of the characteristic equation

$$|E\lambda^2 + D\lambda + F| = 0 \tag{1.3}$$

equals the number of negative eigenvalues of matrix F.